# Theory X and Geometric Representation Theory IV

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Let's start by explaining more compactifications of $X_{\mathfrak{g}}$. The theories $X_{\mathfrak{g}}[M^3]$ is a class of theories called Class R, studied by Tudor and coauthors. The further reduction $X_{\mathfrak{g}}[M^3 \times S^1]$ is the Landau-Ginzburg model associated to Chern-Simons for $G$, and hence $X_{\mathfrak{g}}[M^3 \times T^2]$ is the Jacobi ring of Chern-Simons. $X_{\mathfrak{g}}[D^2 \times S^1]$ is Chern-Simons for $G$, hence we know its further reductions:

1. $X_{\mathfrak{g}}[D^2 \times T^2]$ is the modular tensor category Chern-Simons assigns to a circle (representations of the loop group or the quantum group), or equivalently the $G/G$-gauged WZW model.
2. $X_{\mathfrak{g}}[D^2 \times S^1 \times \Sigma]$ is the space of WZW conformal blocks for $\Sigma$, and
3. $X_{\mathfrak{g}}[D^2 \times S^1 \times M^3]$ is the Chern-Simons invariant of $M$.

$X_{\mathfrak{g}}[S^1 \times \Sigma]$ is Rozansky-Witten theory with target the Hitchin moduli space, so $X_{\mathfrak{g}}[S^2 \times S^1 \times \Sigma]$ is functions on $\mathrm{Loc}_G(\Sigma)$.

Recall that if $Z$ is an $n$-dimensional TFT then $Z(S^{n-1})$ is an $E_n$ algebra of local operators, coming from putting small balls inside a bigger ball. In the setting of chain complexes we can think of as functions on a space with a Poisson bracket of degree $1-n$.

Similarly, with suitable targets, $Z(S^{n-2})$ is an $E_{n-1}$ category. More generally, $Z(S^{n-k})$ is an $E_{n-k+1}$ $(k-1)$-category. The two numbers involved here have to add up to $n$.

In Rozansky-Witten theory with target a holomorphic symplectic $M$ we saw that $Z(S^2)$ is $O(M)$. It turns out that

$$Z(S^1) = \mathrm{DCoh}(M) \tag{1}$$

should be the derived category of coherent sheaves. Similarly, if $Z = X_{\mathfrak{g}}[\Sigma]$, then when $\Sigma = T^2$, $Z(S^1)$ should be the spherical Hecke category.

What is the $E_2$ structure on $\mathrm{DCoh}(M)$? One might think that it's the tensor product of coherent sheaves, but $E_2$ means braided monoidal, so that isn't quite right. The actual structure is a deformation of the tensor product.

Consider the special case $M = T^*X$, or look near a Lagrangian. Then we have a Koszul duality

$$\mathrm{DCoh}(T^*X) \cong \mathrm{DCoh}(TX[-1]) \tag{2}$$

(where we're only remembering $\mathbb{Z}_2$ gradings) which is in turn $\mathrm{DCoh}(LX)$ where $LX$ is the derived loop space. Now there is an obvious thing to do, which is to convolve using the natural $E_2$ structure on $LX$. Explicitly, there is a span from $LX \times LX$ to $LX$ given by the formal space of maps from a pair of pants into $X$, and we do a push-pull transform on this.

There is an $S^1$ action on $\mathrm{DCoh}(LX)$ corresponding to the de Rham differential. It induces a natural deformation coming from working $S^1$-equivariantly, and the corresponding deformation of $\mathrm{DCoh}(T^*X)$ is D-modules $D_X$-Mod on $X$ (actually a funny graded version). This is a version of deformation quantization.

Q: do we know a good global version of this construction, without looking near a Lagrangian?

A: not that I know of. Fortunately the spaces we care about are already cotangent bundles.

If $A$ is an $E_n$ algebra, then $\mathrm{Mod}(A)$ is naturally an $E_{n-1}$ category. A degenerate example: when $n = 1$, then if $A$ is an algebra, $\mathrm{Mod}(A)$ is an $E_0$ category, which means that it has a distinguished object, namely $A$. Or when $n = \infty$, then if $A$ is a commutative algebra, $\mathrm{Mod}(A)$ is a symmetric monoidal category.

Inside of $Z(S^1)$ there is a distinguished object which is the unit of the $E_2$ algebra structure, which is $Z(D^2)$. More generally, $Z(D^{k+1})$ is the unit of the $E_{k+1}$ algebra structure on $Z(S^k)$. Hence

$$Z(S^2) = Z(D^2 \sqcup_{S^1} D^2) = \mathrm{End}_{Z(S^1)}(1). \tag{3}$$

More generally, in an oriented setting, $Z(S^k) = \mathrm{End}_{Z(S^{k-1})}(1)$. In our setting, the unit object 1 for $\mathrm{DCoh}(M)$ is the structure sheaf $O$, and $Z(S^2)$ is (derived) global sections $O(M)$ of the structure sheaf. This loses information if $M$ is not affine. As a slogan, suspension is affinization.

Recall that an $E_1$ algebra $A$ is the same thing as a category with one object and endomorphisms given by $A$. We can also think of it as presenting a category generated by this thing, getting us $\mathrm{Mod}(A)$ (right modules). Similarly, if $A$ is an $E_2$ algebra, then it is the same thing as a 2-category with one object and one morphism with endomorphisms given by $A$. We can also think of it as presenting a 2-category generated by this thing, getting us $\mathrm{Mod}(\mathrm{Mod}(A))$. Thinking this way suggests, for example, that the natural notion of map between $E_1$ algebras is a functor between their module categories; this is Morita theory.

$Z(S^k)$ is an $E_{k+1}$ $(n-k-1)$-category, but by iterating the above construction we can think of this as just presenting an $n$-category generated by morphisms at various category levels.

Our first guess at the moduli space of $Z$ is

$$\mathrm{Spec}\, Z(S^{n-1}) \tag{4}$$

where $Z(S^{n-1})$ is an $E_n$ algebra. But we can do better! $Z(S^{n-2})$ is an $E_{n-1}$ category with a distinguished object $1 = Z(D^{n-1})$ with $\mathrm{End}(1) = Z(S^{n-1})$, so there is a natural map

$$\mathrm{Mod}(Z(S^{n-1})) \to Z(S^{n-2}) \tag{5}$$

of $E_{n-1}$ categories. For example, when $Z(S^1) = \mathrm{DCoh}(M)$, then the inclusion of the inclusion of modules over $Z(S^2)$ is modules over global sections $O(M)$. So corresponding to the tower of algebras associated to the spheres is a tower of moduli spaces

$$\cdots \to \mathrm{Spec}\, Z(S^{n-3}) \to \mathrm{Spec}\, Z(S^{n-2}) \to \mathrm{Spec}\, Z(S^{n-1}) \tag{6}$$

where the claim is that each of these maps is a kind of affinization. In other words, by using more than local operators, we see more of the moduli space.

In Rozansky-Witten theory on $M$, $\operatorname{Spec} Z(S^1)$ is the thing we want. Recall that $\operatorname{Spec} Z(S^2)$ only detects the affinization of $M$. But $Z(S^1) = \operatorname{DCoh}(M)$ with its tensor product knows all of $M$ (where for now we're ignoring the subtlety with the braided monoidal structure); this is precisely Tannakian reconstruction for geometric stacks. The idea is that if $M$ is such a thing, then

$$M = \operatorname{Spec} \operatorname{QCoh}(M) \tag{7}$$

where we remember the tensor structure on $\operatorname{QCoh}(M)$. What do we mean by this? We'll describe it by describing its functor of points: if $R$ is any ring and if $C$ is a suitable tensor category, then

$$(\operatorname{Spec} C)(R) = \operatorname{Hom}_{\otimes}(C, \operatorname{Mod}(R)). \tag{8}$$

This is not a tautological Yoneda argument because we are only looking at particular tensor categories of the form $\operatorname{Mod}(R)$. For a general tensor category $C$, there is a comparison map

$$C \to \operatorname{QCoh}(\operatorname{Spec} C) \tag{9}$$

and it won't be an isomorphism in general. It will be in our examples.

**Example** In Class S theories $Z = X_{\mathfrak{g}}[\Sigma]$, our first approximation of the moduli space is

$$\operatorname{Spec} Z(S^3) = B \tag{10}$$

which is the Hitchin base. Our second approximation is

$$\operatorname{Spec} Z(S^2) = M \tag{11}$$

which is the Hitchin space itself.

Similarly in Rozansky-Witten theory. In general, if $C$ is an $E_{n-1}$ category, then the prescription

$$(\operatorname{Spec} C)(R) = \operatorname{Hom}_{\otimes}(C, \operatorname{Mod}(R)) \tag{12}$$

makes sense when $R$ is an $E_n$ algebra, so $C$ has a functor of points which takes $E_n$ algebras as input.

In more physical language, $Z(S^{n-1})$ is local operators (since we can delete a small ball in any $n$-manifold and evaluate the result on $Z(S^{n-1})$). Similarly, $Z(S^{n-2})$ is line operators, and so forth; this should all be understood in the context of the cobordism hypothesis with singularities. The cobordism hypothesis with singularities tells us that if $Z(S^{n-2})$ (which is not symmetric monoidal, so we can't directly apply the ordinary cobordism hypothesis) has a suitably dualizable object in it, then we can write down a 1d field theory based on it relative to the bulk theory $Z$.

So the above construction corresponds to detecting the moduli space not just with local operators, but also with line operators, etc. Where local operators give functions on the moduli space, line operators give vector bundles or sheaves on the moduli space, etc.

A theory $Z$ is linear over $Z(S^k)$ for all $k$ in a suitable sense. Roughly speaking,

$$Z(M) \in \int_M Z(S^k). \tag{13}$$

What does this mean? Let's think of $Z(S^k)$ as an $n$-category, which is the sort of thing we'd like to assign an $(n+1)$-dimensional field theory to. So $Z$ looks like a B-model into the $Z(S^k)$ version of the moduli space. The simplest version of this statement

$$Z(M) \in \int_M Z(S^{n-1}) \tag{14}$$

can be made sense of by interpreting the RHS as factorization homology of the $E_n$ algebra $Z(S^{n-1})$.

Now it's time to explain geometric Langlands. Consider the dimensional reduction $Z = X_{\mathfrak{g}}[\Sigma]$. The moduli space we want is

$$\widetilde{M} = \operatorname{Spec} Z_{S^1}(S^1) = \operatorname{Spec} Z(S^1 \times S^1). \tag{15}$$

This is fibered over $B = \operatorname{Spec} Z(S^3)$. There are two monoidal structures on $Z(S^1 \times S^1)$ (in fact an $\mathrm{SL}_2(\mathbb{Z})$ worth) coming from the different circles showing that $\widetilde{M} \to B$ is a family of group schemes which is Ngo's variant of the Hitchin fibration, or the Seiberg-Witten integrable system of a TFT.

We can also work equivariantly with respect to both circles. There is both a deformation quantization and a commutative deformation corresponding to a quantization and a deformation of the Hitchin space, which are switched by $\mathrm{SL}_2(\mathbb{Z})$. This is geometric Langlands.

This all comes out of discussions with Andrew Neitzke, Thomas Nevins, and David Nadler.